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VARIABLE DENSITY APPROACH FOR ROTATING SHALLOW SHELL OF VARIABLE THICKNESS

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Abstract—A new approach based on variable density in conjunction with shallow shell theory is proposed to analyse rotating shallow shell of variable thickness. Coupled non-linear ordinary differential equations governing shallow shells of variable thickness are first derived before applying the variable density approach. Results obtained from the new approach compare well with FEM calculation for a wide range of profiles considered in this paper.

NOTATION

E	Young's modulus
ν	Poisson's ratio
a	radius of shallow shell
r_c	maximum radial distance from axis of rotation
r	radial distance from axis of rotation
ω	angular velocity in radian s^{-1}
p	load intensity in normal direction $= \rho\omega^2 hr^2/a^2$
p_r	load intensity in meridional direction $\approx \rho\omega^2 hr(1-(r^2/2a^2))$
Ω	body force potential $= -\int p_r \, dr$
F	stress function
Δ	$(d^2/dr^2)+(1/r)(d/dr)$ Laplacian
l	characteristic length $= \sqrt{ah}/\sqrt[4]{12(1-\nu^2)}$
x	dimensionless quantity $= r/l$
D	$Eh^3/12(1-\nu^2)$
ε_r	radial strain
ε_θ	tangential strain
δ	horizontal displacement
v	displacement in meridional direction
N_r	membrane force in radial direction
N_θ	membrane force in tangential direction
σ_c	constant stress
h	thickness of the rotating shell $h' = dh/dr$, $h'' = d^2h/dr^2$
w	normal deflection $w' = dw/dr$, $w'' = d^2w/dr^2$
ρ	mass density at radius r
M_r	bending moment per unit length in radial direction
M_θ	bending moment per unit length in tangential direction
Q_r	shearing force in radial direction
Q_v	vertical shearing force.

INTRODUCTION

Shallow rotating shells are expected to play a major role in future space missions and communication technology when large spinning disks and antennas are deployed to provide for propulsion by solar energy, transmit data, or perform other maneuvers. Regarding applications on ground, increasing power and speed of rotating machinery demand faster techniques of theoretical stress analysis to achieve optimal designs before FEM implementation on prototypes. It is in this context that variable thickness shells promise unexplored avenues in engineering design. It is well known that the analysis of a rotating shallow shell becomes very complicated when its thickness varies. The corresponding problem of rotating disks of variable thickness has been extensively investigated culminating in the design and development of high speed turbine and compressor disks. The idea behind variable thickness disks is in evolving a disk of constant strength in which inplane stresses remain fixed throughout the disk. Accordingly, as far back as 1927, Stodola (1927) suggested a hyperbolic profile to achieve constant strength. Further development of this concept followed on the lines of Donath (1912), Hearle (1918) and Grammel (1936). A general solution to the problem of a disk with a polynomial variation in its thickness is treated widely in the literature.

The problem of a rotating shallow shell with variable thickness differs considerably from that of a rotating disk due to the bending action. Shell theories have been successfully developed to tackle several structural problems to resist mechanical, thermal, gravitation and centrifugal loads for constant wall thickness (Flügge, 1973; Lin and Wan, 1985). Spherical domes and cylindrical tanks of variable thickness were also examined to optimize stress levels and reduce weight. The general theory of variable thickness shells as well as an approximation for thin shells is extensively discussed in Flügge (1973). An exact solution for the general case of variable thickness shells is formidable. On the other hand, numerical solutions mask the role of parameters in engineering design. This paper presents a viable solution for the problem of a rotating shallow shell with variable thickness by assuming variable density. Results are presented for radial and tangential bending moment and membrane force for different cases of thickness variation. Results are also compared with FEM calculation performed on the variable thickness shell to examine the accuracy of the present approach.

VARIABLE THICKNESS FORMULATION

The actual shallow shell of variable thickness is shown in Fig. 1(a). For the sake of completeness we derive the governing equations for a variable thickness shell before taking

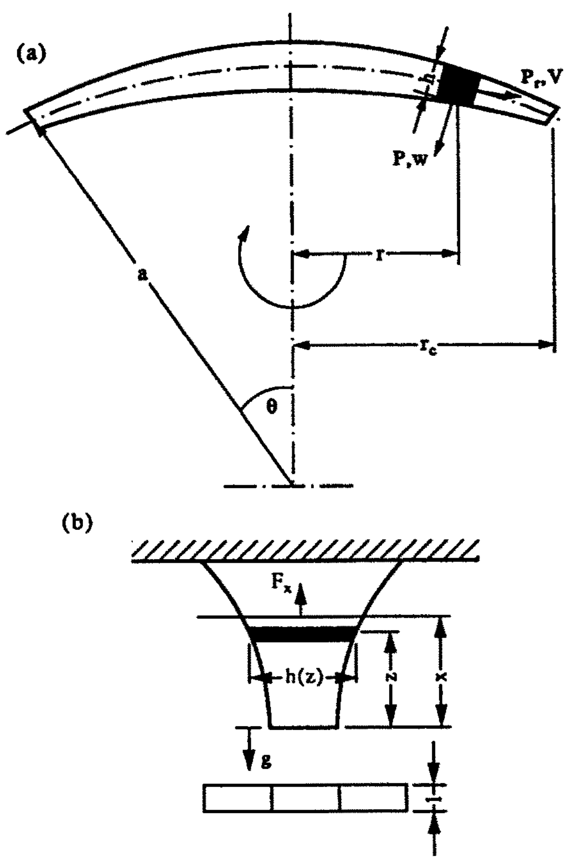


Fig. 1. (a) Shallow disk with variable thickness. (b) Force at a distance x due to gravity loading. $F_x = \int_0^x \rho g h(z)\,dz$. (c) Force at a distance x due to rotating bar. $F_x = \int_0^x \rho\omega^2 h(z)(x-z)\,dz$. (d) Shear force V_x and bending moment M_x due to gravity loading at a distance x, $V_x = \int_0^x \rho g h(z)\,dz M_x = \int_0^x \rho g h(z)(x-z)\,dz$. (e) Shallow disk with uniform thickness.

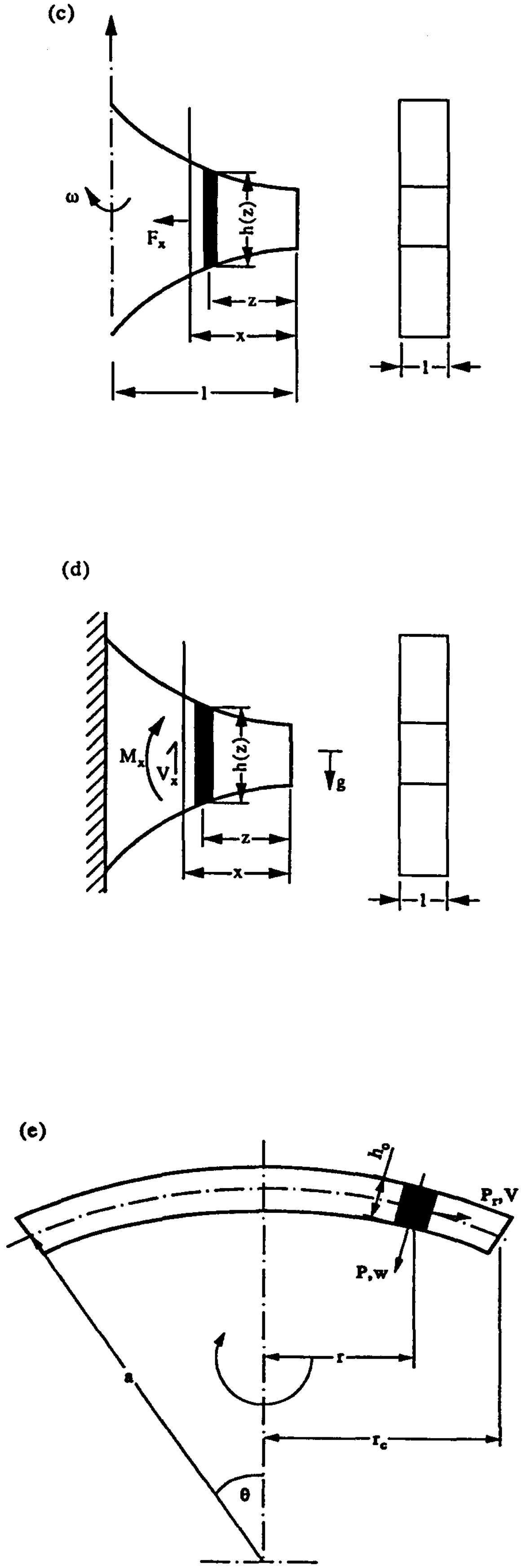


Fig. 1. *Continued.*

up the case of variable density approach. We follow the general procedure outlined in Flügge (1973), and Timoshenko and Woinowosky–Krieger (1989).

The differential equations of equilibrium are

$$\frac{d(rN_r)}{dr} - N_\theta - \frac{r}{a}Q_r + rp_r = 0, \tag{1}$$

$$\frac{d(rQ_r)}{dr} + \frac{r}{a}(N_r + N_\theta) + rp = 0, \tag{2}$$

$$\frac{d(rM_r)}{dr} - M_\theta - rQ_r = 0. \tag{3}$$

The strains are:

$$\varepsilon_r = \frac{1}{Eh}(N_r - \nu N_\theta) = \frac{dv}{dr} - \frac{w}{a},$$

$$\varepsilon_\theta = \frac{1}{Eh}(N_\theta - \nu N_r) = \frac{v}{r} - \frac{w}{a}. \tag{4}$$

The bending moments are:

$$M_r = -D(\chi_r + \nu\chi_\theta) = -D\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right),$$

$$M_\theta = -D(\chi_\theta + \nu\chi_r) = -D\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right). \tag{5}$$

Assuming $p_r = -d\Omega/dr$, Ω representing a radial body force potential, the force resultants per unit length are:

$$N_r = \frac{1}{r}\frac{dF}{dr} + \Omega,$$

$$N_\theta = \frac{d^2F}{dr^2} + \Omega. \tag{6}$$

Using (4) the compatibility equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varepsilon_\theta}{dr}\right) - \frac{1}{r}\frac{d\varepsilon_r}{dr} + \frac{1}{a}\Delta w = 0. \tag{7}$$

Combining (4) and (7) we arrive at the following fundamental equation for F and w:

$$\Delta\Delta F + \frac{Eh}{a}\Delta w - \frac{h'}{h}\left[2F''' + \frac{(2-\nu)}{r}F'' - \frac{F'}{r^2}\right] - \frac{h''}{h}\left(F'' - \frac{\nu}{r}F'\right) + 2\left(\frac{h'}{h}\right)^2\left(F'' - \frac{\nu}{r}F'\right)$$

$$= -(1-\nu)\Delta\Omega + \frac{h'}{h}\left[2(1-\nu)\Omega' + (1-\nu)\frac{\Omega}{r}\right] + \frac{h''}{h}(1-\nu)\Omega - 2\left(\frac{h'}{h}\right)^2(1-\nu)\Omega. \tag{8}$$

The second fundamental relation between F and w is obtained by substituting Q_r from (3) in (2):

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{\mathrm{d}(rM_r)}{\mathrm{d}r}-M_\theta\right]+\frac{r}{a}(N_r+N_\theta)+rp=0. \tag{9}$$

Using (5) and (6) in combination with (9) gives

$$\Delta\Delta w-\frac{1}{aD}\Delta F+\frac{3h'}{h}\left[2w'''+\frac{(2+\nu)}{r}w''-\frac{w'}{r^2}\right]+\frac{3h''}{h}\left(w''+\frac{\nu}{r}w'\right)$$
$$+6\left(\frac{h'}{h}\right)^2\left(w''+\frac{\nu}{r}w'\right)=\frac{2\Omega}{aD}+\frac{p}{D}. \tag{10}$$

VARIABLE DENSITY APPROACH

The solution of simultaneous equations (8) and (10) is formidable. Shells of variable thickness seldom admit exact analysis although plates of variable thickness admit some exact theory within the context of small deflection plate theory. Timoshenko and Woinowosky–Krieger (1989) have presented a large number of detailed solutions to many variable thickness plate problems, and it is not the intention here to duplicate the work but rather to present a different approach to tackle the rotating shallow shell problem. With regard to shells of variable thickness, Flügge (1973) has discussed the design of constant strength dome under its own weight. Centrifugal loading represents another example of body force problems. Material density enters an important factor in problems dominated by body forces. With particular reference to beams, plates and shells of variable thickness quite often it is found that force, moment resultants depend on the product of density and thickness. This is illustrated with a few examples for the case of bar or a beam (Figs 1(b)–(d)). In the above problems we can treat the product of density and thickness as a single variable to the extent of calculating force and moment resultants. The purpose of this paper is to examine the extension of this concept to two-dimensional structures such as plates and shells. However, it should be borne in mind that the final calculation of stresses requires the actual thickness at a particular section. Further, the present approach of lumping density and thickness in a single variable may lead to discrepancies in displacements as will be shown in this paper.

With the above prelude we treat the problem of a rotating shell of variable thickness as a shell of constant thickness but variable density, as shown in Fig. 1(a), is replaced by a shell of variable density (Fig. 1(e)) such that the total mass of the shell is conserved. The equivalent shell of variable density also ensures the same mass variation with radius and hence the same moment of inertia. The thickness variation is assumed to follow an exponential profile $h=h_0\,\mathrm{e}^{-kr^2}$. It may be recalled that this profile gives a rotating disk of constant strength; the constant $k=\rho\omega^2/2\sigma_c$ involves disk density, speed and stress level. It is interesting to note here that this profile also approximates a shallow dome of constant strength discussed in Flügge (1973) where $k=-\rho g/2a\sigma_c$. However, in the case of constant strength dome, the shell thickness increases with r to give a constant compressive membrane stress σ_c; thickness decreases in the case of a rotating shallow shell to achieve a uniform tensile stress σ_c induced by the centrifugal action.

Variable density approach simplifies the problem enormously by assuming constant thickness. Since $h'=h''=0$, (8) and (10) reduce to:

$$\Delta\Delta F+\frac{Eh}{a}\Delta w=-(1-\nu)\Delta\Omega, \tag{11}$$

$$\Delta\Delta w-\frac{1}{Da}\Delta F=\frac{p}{D}+\frac{2\Omega}{Da}. \tag{12}$$

Integration of simultaneous equations (11) and (12) can be carried out by multiplying eqn (11) by a factor $-\lambda$ and adding the result to eqn (12). This yields

$$\Delta\Delta(w-\lambda F)-\lambda(Eh/a)\Delta(w+F/\lambda hDE) = \lambda(1-\nu)\Delta\Omega+\frac{p}{D}+\frac{2\Omega}{Da}. \tag{13}$$

HOMOGENEOUS SOLUTION ($p = \Omega = 0$)

Stipulating $\lambda = -1/\lambda hDE$ yields

$$\Delta\Delta\psi-\frac{\lambda Eh}{a}\Delta\psi = 0,$$

where $\psi = w-\lambda F$ and $\lambda = (i)/(Eh^2)\sqrt{12(1-\nu^2)}$. The solution for the above equation is $\psi = \psi_1+\psi_2$, where

$$\psi_1 = A_1+A_2\log x, \tag{14}$$

$$\psi_2 = A_3[\mathrm{Ber}\,(x)+i\,\mathrm{Bei}\,(x)]+A_4[\mathrm{Ker}\,(x)+i\,\mathrm{Kei}\,(x)]. \tag{15}$$

The functions Ber, Bei, Ker and Kei are Bessel's functions for imaginary arguments (McLachlan 1955).

In equations (14) and (15), letting $A_i = a_i+ib_i$ where a_i, b_i are real constants to be determined, and comparing real and imaginary parts, we get by recalling $\psi = w-\lambda F$:

$$w = a_1+a_2\log x+a_3\,\mathrm{Ber}\,(x)-b_3\,\mathrm{Bei}\,(x)+a_4\,\mathrm{Ker}\,(x)-b_4\,\mathrm{Kei}\,(x), \tag{16}$$

$$F = -\frac{Eh^2}{\sqrt{12(1-\nu^2)}}[b_1+b_2\log x+a_3\,\mathrm{Bei}\,(x)-b_3\,\mathrm{Ber}\,(x)+a_4\,\mathrm{Kei}\,(x)-b_4\,\mathrm{Ker}\,(x)]. \tag{17}$$

Of the eight constants appearing above, only a_1, a_3 and b_3 control the stress distribution in a solid rotating shell. Since w, N_r are finite at $r = 0$, $a_2 = a_4 = b_2 = b_4 = 0$. Also b_1 can be omitted as it will not produce any stress.

PARTICULAR SOLUTION

In the case $\Omega \neq 0, p \neq 0$ solution of eqn (13) will have a particular integral in addition to a complementary solution. Substituting $\rho = \rho_0\,\mathrm{e}^{-kr^2}$, Ω can be approximated as:

$$\Omega = -\rho_0\omega^2h\left[\frac{r^2}{2}-\left(\frac{k}{4}+\frac{1}{8a^2}\right)r^4+\left(\frac{k^2}{12}+\frac{k}{12a^2}\right)r^6-\frac{k^2r^8}{32a^2}\right]. \tag{18}$$

To obtain the particular solution, we assume

$$w = Q_1r^2+Q_2r^4+Q_3r^6+Q_4r^8+Q_5r^{10}, \tag{19}$$

$$F = -R_1r^2-R_2r^4-R_3r^6-R_4r^8-R_5r^{10}. \tag{20}$$

where $Q_1, \ldots, Q_5$ and $R_1, \ldots, R_5$ are real constants. Substituting the above particular solution into eqn (13), comparing real and imaginary parts, we get the following relations connecting Q_i and R_i

$$Q_1 = (64R_2+2\rho\omega^2h(1-\nu))\frac{a}{4Eh},$$

$$Q_2 = \left(576R_3-\rho\omega^2h\left(4k+\frac{2}{a^2}\right)(1-\nu)\right)\frac{a}{16Eh},$$

$$Q_3 = \left(2304R_4+\rho\omega^2 h\left(3k^2+\frac{3k}{a^2}\right)(1-\nu)\right)\frac{a}{36Eh},$$

$$Q_4 = \left(6400R_5-2\rho\omega^2 h(1-\nu)\frac{k^2}{a^2}\right)\frac{a}{64Eh},$$

$$Q_5 = 0,$$

$$R_1 = -\frac{16Ehl^4Q_2}{a},$$

$$R_2 = -\left(576Q_3+2\rho_0\omega^2\frac{h}{aD}\right)\frac{Ehl^4}{16a},$$

$$R_3 = \left(\rho_0\omega^2 h\left(\frac{3k}{2aD}+\frac{1}{4a^3D}\right)-2304Q_4\right)\frac{Ehl^4}{36a},$$

$$R_4 = -\rho_0\omega^2 h\left(\frac{2k^2}{3aD}+\frac{K}{6a^3D}\right)\frac{Ehl^4}{64a},$$

$$R_5 = \frac{\rho_0\omega^2 Eh^2 l^4 k^2}{1600a^4 D}.$$

COMPLETE SOLUTION

This is obtained by adding eqns (19) and (20) to (16) and (17), respectively,

$$w = a_1+a_3\,\mathrm{Ber}\,(x)-b_3\,\mathrm{Bei}\,(x)+Q_1r^2+Q_2r^4+Q_3r^6+Q_4r^8+Q_5r^{10},$$

$$F = -\frac{Eh^2}{\sqrt{12(1-\nu^2)}}[a_3\,\mathrm{Bei}\,(x)+b_3\,\mathrm{Ber}\,(x)]-R_1r^2-R_2r^4-R_3r^6-R_4r^8-R_5r^{10},$$

$$N_r = -\frac{Eh}{ax}(a_3\,\mathrm{Bei}'\,(x)+b_3\,\mathrm{Ber}'\,(x))-2R_1-4R_2r^2-6R_3r^4-8R_4r^6-10R_5r^8+\Omega,$$

$$M_r = -D\left\{\frac{a_3}{l^2}[\mathrm{Ber}''\,(x)+\frac{\nu}{x}\mathrm{Ber}'\,(x)]-\frac{b_3}{l^2}\left[\mathrm{Bei}''\,(x)+\frac{\nu}{x}\mathrm{Bei}'\,(x)\right]+2Q_1(1+\nu)\right.$$
$$\left.+4Q_2(3+\nu)r^2+6Q_3(5+\nu)r^4+8Q_4(7+\nu)r^6+10Q_5(9+\nu)r^8\right\}.$$

Applying boundary condition $M_r = 0$, $N_r = 0$ at $r = r_c$ or $x = x_c$ determines a_1, a_3, and b_3:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & \mathrm{Bei}'\,(x_c)/x_c & \mathrm{Ber}'(x_c)/x_c \\ 0 & \mathrm{Ber}''\,(x_c)+\dfrac{\nu}{x_c}\mathrm{Ber}'\,(x_c) & -\mathrm{Bei}''\,(x_c)-\dfrac{\nu}{x_c}\mathrm{Bei}'\,(x_c) \end{bmatrix}\begin{bmatrix} a_1 \\ a_3 \\ b_3 \end{bmatrix}$$
$$= -\begin{bmatrix} 0 \\ \dfrac{ax_c}{Eh}[2R_1+4R_2r_c^2+6R_3r_c^4+8R_4r_c^6+10R_5r_c^8-\Omega] \\ 2Q_1(1+\nu)+4Q_2(3+\nu)r_c^2+6Q_3(5+\nu)r_c^4+8Q_4(7+\nu)r_c^6+10Q_5(9+\nu)r_c^8 \end{bmatrix}. \quad (21)$$

Table 1.

Sl. No.	$k \times 10^6$	Tip thickness (mm)
1	0.0	10
2	2.634	9
3	5.5786	8
4	8.9169	7
5	12.771	6
6	17.329	5

RESULTS

Results of the present theory are tested on a representative rotating shallow shell made of steel (specific gravity = 7.85). Rotation, curvature, tip radius and shell depth are held constant at 17,000 rpm, 1000 mm, 200 mm and 15.38 mm, respectively. The tip thickness is varied from 10 mm (constant thickness case) to 5 mm by changing the value of k listed in Table 1. As mentioned earlier, the profile chosen for the study gives a constant strength disk in the absence of curvature of approximately $\sigma_c = 80$ kg mm^{-2} (784 Mpa). This stress level corresponds to a membrane force of 800 kg mm^{-1} for a shell thickness of 10 mm. Other data relating to dimensions and speed are based on existing designs of prototypes and photoelastic models. FEM calculations were performed using a general purpose commercial software. Eight noded axisymmetric solid elements were used to examine the rotating shell of variable thickness. For the case of uniform shell thickness, FEM results matched well with exact theory validating the overall FEM procedure followed in this investigation.

For the first case of constant shell thickness, the normal deflection w, radial and tangential membrane force and bending moment are displayed in Figs 2(a–c). The maximum tip displacement of 5.53 mm according to theory compares well with FEM value of 5.74 mm. However, the stresses agree even better to within 2%. It is also worth noting that the tangential membrane force is well within the limit of 800 kg mm^{-1} assigned for the uniform strength disk Fig. 2(b). Also shown in this figure is the stress variation in a constant thickness disk. In this case, both radial and hoop stresses decrease towards the tip. In the rotating shallow shell, the hoop stress increases with radius for the dimensions selected in this study.

To demonstrate the variable density approach, consider the results shown in Figs 3(a–c) for the case of tip thickness 9 mm. Tip displacement according to variable density approach is 4.91 mm whereas FEM gives 5.57 mm as shown in Fig. 3(a). In spite of this discrepancy, variable density approach predicts N_r and N_θ, M_r and M_θ quite well. This feature continues for all the cases considered here as shown in Figs 3–7. This reveals the usefulness of variable density approach for stress analysis rather than displacements. In general the error does not exceed about 10% for membrane force calculation and 15% for bending moment calculation in the extreme case of thickness variation (case 6). The error magnitude is expected to increase with decreasing tip thickness, and further research is necessary to modify the density variation function to improve the results.

It is interesting to examine the extreme case 6 further. Although the tip thickness has reduced to 5 mm, the maximum tangential membrane force N_θ reaches a value of 400 kg mm^{-1}, or a stress level of 80 kg mm^{-2}. Similarly there is a reduction in the value of M_r and M_θ. There is no significant change in the radial membrane force distribution see Figs 2(b)–7(b). There is a similar increase in tangential bending stress at the tip. The net effect of varying the thickness appears to render the stresses more uniform than in the case of constant thickness shell. The present approach can therefore be employed in optimizing the shape of a rotating disk with small initial curvature.

CONCLUSION

Exact analysis of variable thickness shells is quite complicated. Variable density approach appears to provide a rapid solution to the problem of rotating shallow shells of

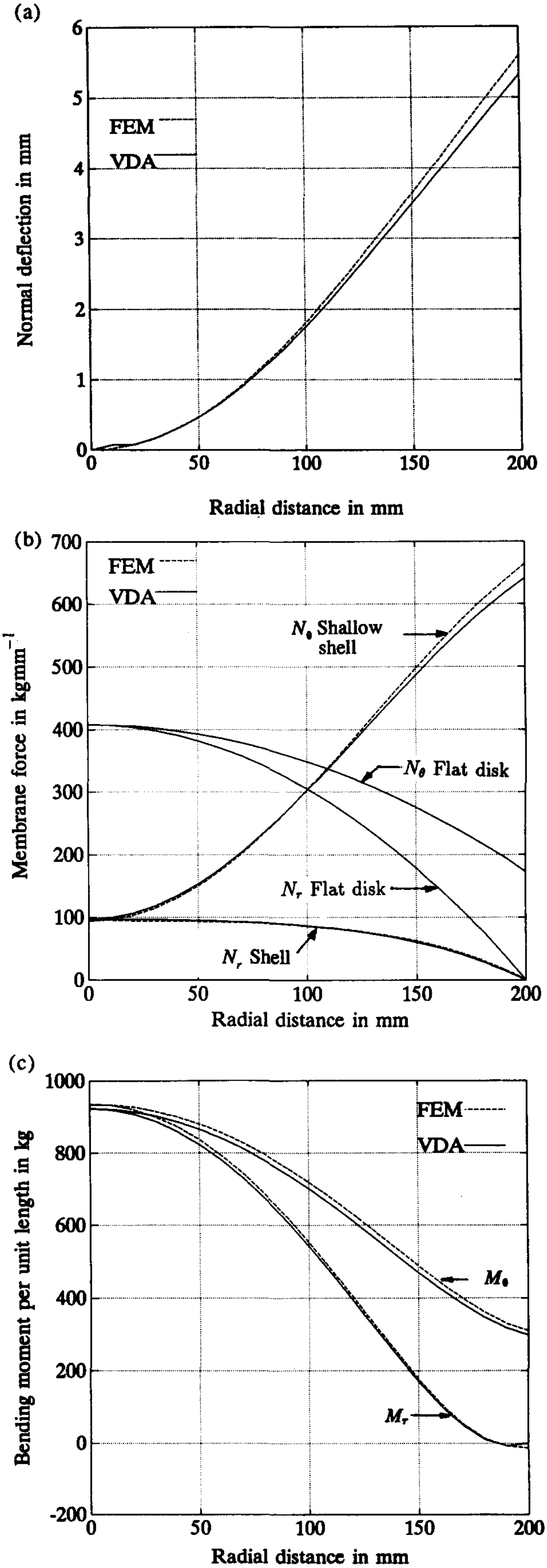


Fig. 2. (a) Comparison of normal deflection between FEM and variable density approach in (case 1). (b) Variation of radial and tangential membrane force for shallow shell and flat disk in (case 1). (c) Comparison of bending moment per unit length between FEM and variable density approach in (case 1).

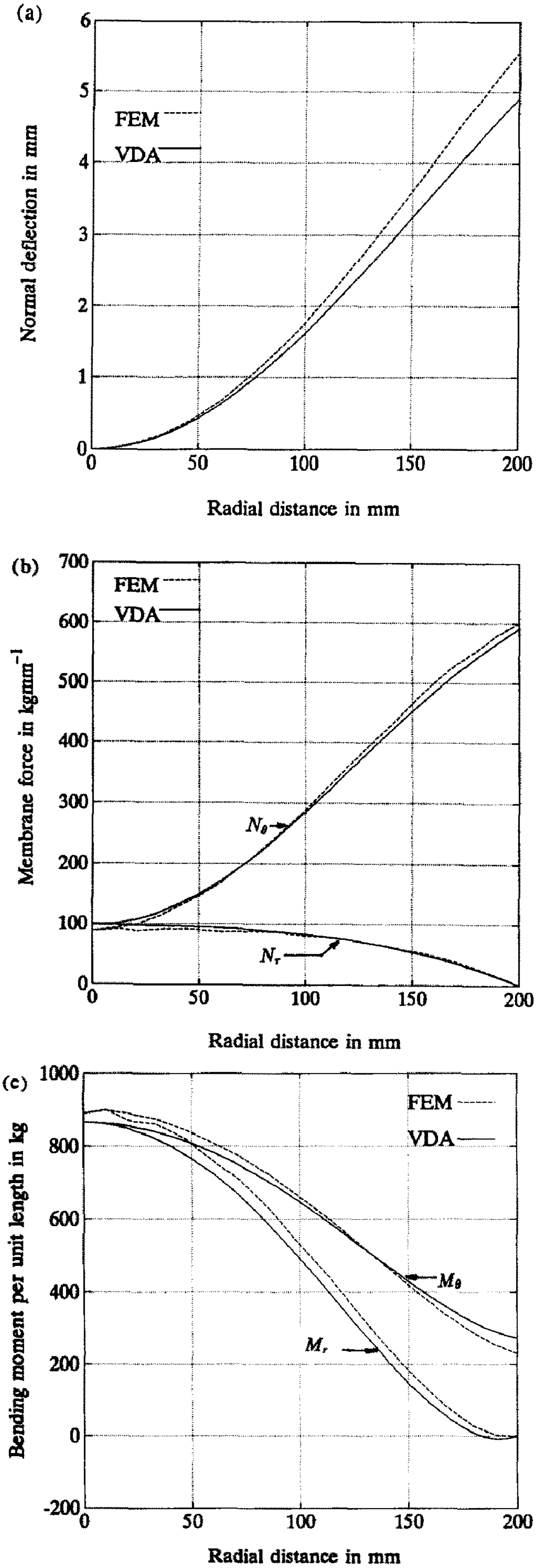


Fig. 3. (a) Comparison of normal deflection between FEM and variable density approach in (case 2). (b) Comparison of radial and tangential membrane force between FEM and variable density approach in (case 2). Comparison of bending moment per unit length between FEM and variable density approach in (case 2).

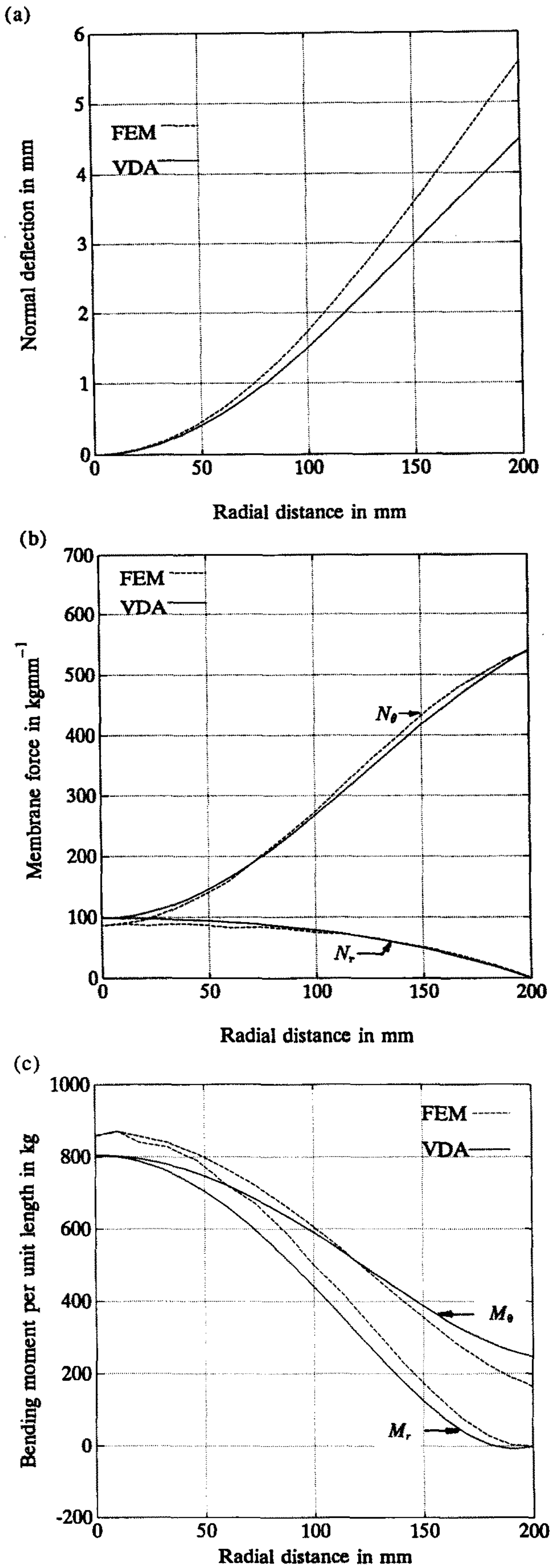


Fig. 4. (a) Comparison of normal deflection between FEM and variable density approach in (case 3). (b) Comparison of radial and tangential membrane force between FEM and variable density approach in (case 3). (c) Comparison of bending moment per unit length between FEM and variable density approach in (case 3).

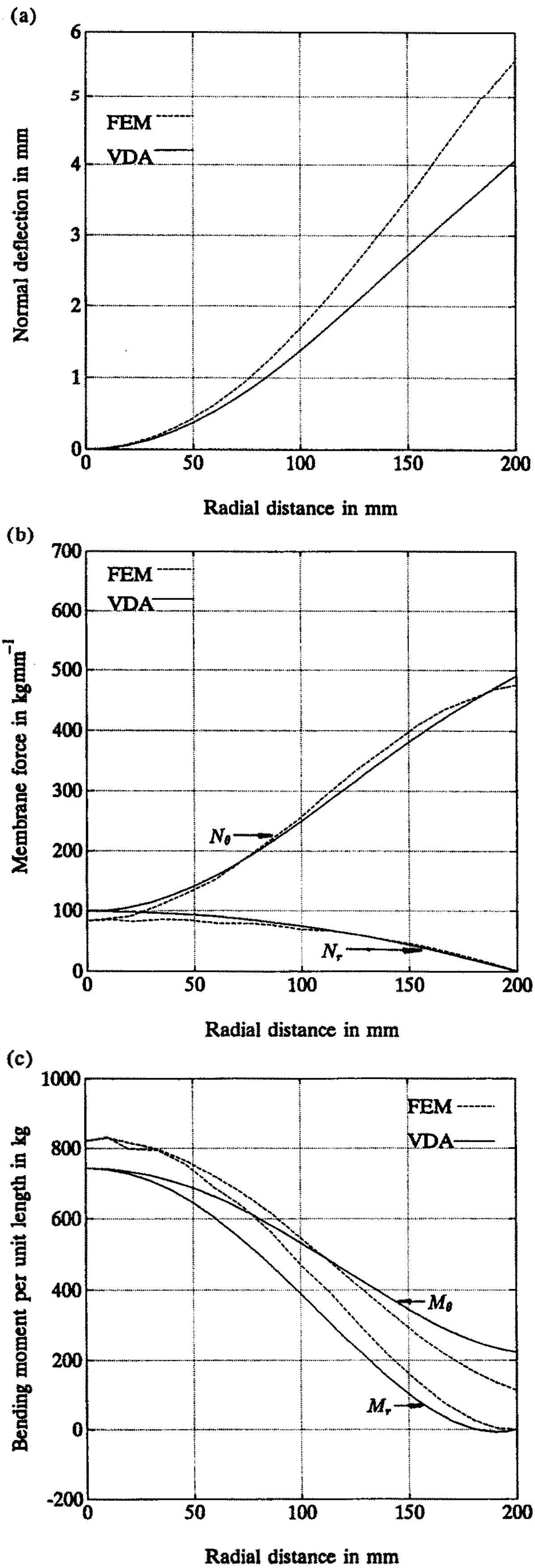


Fig. 5. (a) Comparison of normal deflection between FEM and variable density approach in (case 4). (b) Comparison of radial and tangential membrane force between FEM and variable density approach in (case 4). (c) Comparison of bending moment per unit length between FEM and variable density approach in (case 4).

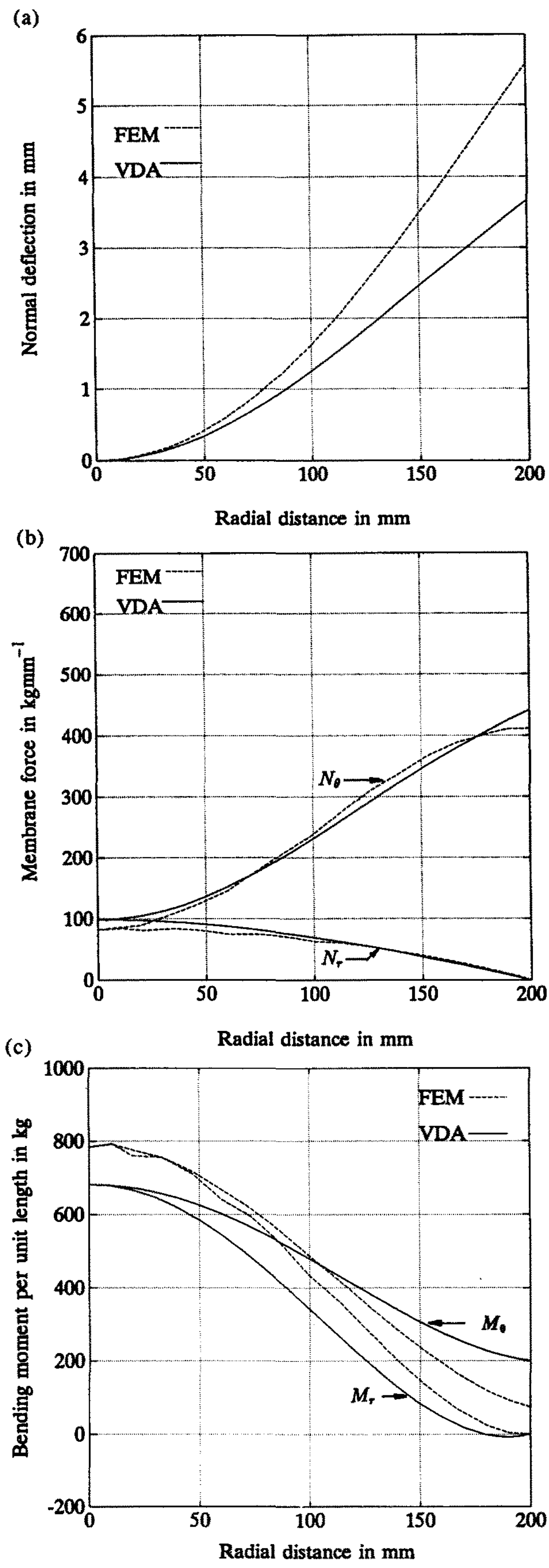
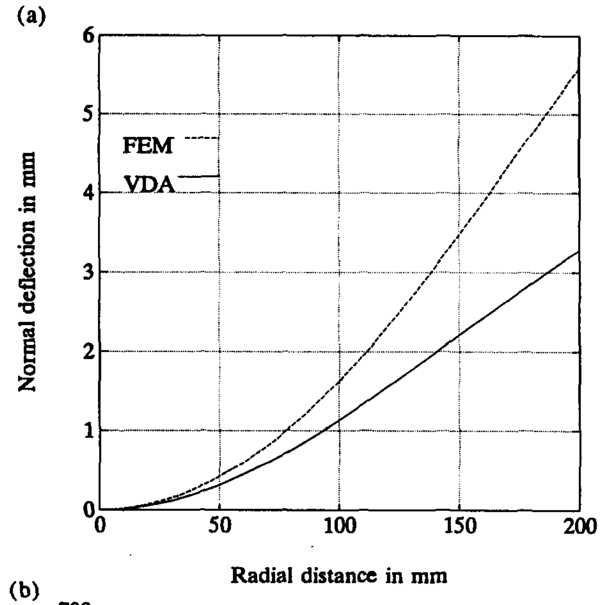
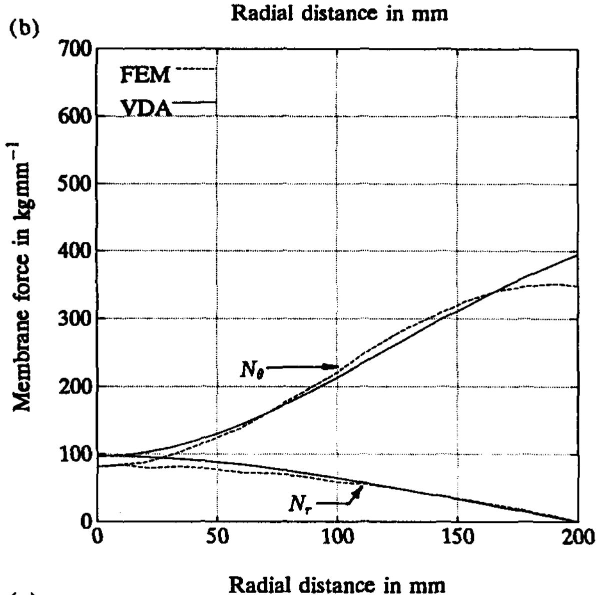


Fig. 6. (a) Comparison of normal deflection between FEM and variable density approach in (case 5). (b) Comparison of radial and tangential membrane force between FEM and variable density approach in (case 5). (c) Comparison of bending moment per unit length between FEM and variable density approach in (case 5).





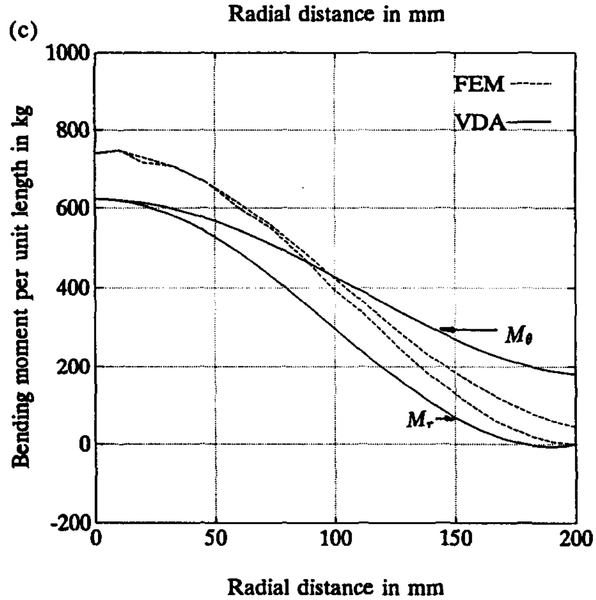


Fig. 7. (a) Comparison of normal deflection between FEM and variable density approach in (case 6). (b) Comparison of radial and tangential membrane force between FEM and variable density approach in (case 6). (c) Comparison of bending moment per unit length between FEM and variable density approach in (case 6).

variable thickness. Particularly with regard to stress analysis, the present methodology can aid in optimizing the shape of rotating discs with initial curvature. It is possible to extract the individual contributions of membrane and bending action to the stresses. Further research is necessary to optimize the variable density function to mimic the effect of variable thickness to improve the results particularly with regard to the displacements. The variable density technique can be easily extended to other profiles than those considered in this paper.

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